

Simultaneous Unitary Equivalences*

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Dedicated with respect and appreciation to Avi Berman, Moshe Goldberg, and Raphael Loewy on the occasion of their retirement from The Technion.

Abstract

Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ be given finite sets of pairs of n -by- n complex matrices. We describe an algorithm to determine, with finitely many computations, whether there is a single unitary matrix U such that each pair of matrices in \mathcal{S}_1 is unitarily similar via U , each pair of matrices in \mathcal{S}_2 is unitarily congruent via U , each pair of matrices in \mathcal{S}_3 is unitarily similar via \bar{U} , and each pair of matrices in \mathcal{S}_4 is unitarily congruent via \bar{U} .

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1 Introduction

Our goal is to solve the following

General Problem. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ be given finite sets of pairs of n -by- n complex matrices. Describe an algorithm to determine, with finitely many

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computations, whether there is a single unitary matrix U such that each pair of matrices in \mathcal{S}_1 is unitarily similar via U , each pair of matrices in \mathcal{S}_2 is unitarily congruent via U , each pair of matrices in \mathcal{S}_3 is unitarily similar via \bar{U} , and each pair of matrices in \mathcal{S}_4 is unitarily congruent via \bar{U} .

This General Problem includes as special cases the problem of determining whether finitely many pairs of matrices are simultaneously unitarily similar ($\mathcal{S}_2 = \mathcal{S}_3 = \mathcal{S}_4 = \emptyset$) as well as the problem of determining whether finitely many pairs of matrices are simultaneously unitarily congruent ($\mathcal{S}_1 = \mathcal{S}_3 = \mathcal{S}_4 = \emptyset$).

All of the matrices that we consider are complex and square. Two matrices A and B of the same size are *unitarily similar* if there is a unitary matrix U such that $A = UBU^*$; they are *unitarily congruent* if there is a unitary matrix U such that $A = UBU^T$. Given pairs of n -by- n matrices $(A_1, B_1), \dots, (A_m, B_m)$ are *simultaneously unitarily similar* if there is a unitary matrix U such that $A_j = UB_jU^*$ for each $j = 1, \dots, m$; they are *simultaneously unitarily congruent* if there is a unitary matrix U such that $A_j = UB_jU^T$ for each $j = 1, \dots, m$. The trace of a matrix A is denoted by $\text{tr } A$. We adopt the notation and terminology of [4].

2 Unitary similarity of a pair of matrices

Any finite formal product of nonnegative powers of two noncommuting variables s, t

$$W(s, t) = s^{m_1}t^{n_1}s^{m_2}t^{n_2} \dots s^{m_k}t^{n_k}, \quad m_1, n_1, \dots, m_k, n_k \geq 0$$

is a *word* in s and t . The sum $m_1 + n_1 + m_2 + n_2 + \dots + m_k + n_k$ is the *length* of the word $W(s, t)$, and the nonnegative integers m_i and n_i are its *factor exponents*. A word in A and A^* is

$$W(A, A^*) = A^{m_1}(A^*)^{n_1}A^{m_2}(A^*)^{n_2} \dots A^{m_k}(A^*)^{n_k} \quad (1)$$

If $A = UBU^*$ for some unitary $U \in M_n$, a calculation reveals that $W(A, A^*) = UW(B, B^*)U^*$, so $W(A, A^*)$ is unitarily similar to $W(B, B^*)$. Thus, unitary similarity of A and B implies that

$$\text{tr } W(A, A^*) = \text{tr } W(B, B^*) \quad (2)$$

for every word $W(s, t)$ in two noncommuting variables.

A theorem of W. Specht [7] provides a converse for this implication. Various authors have provided bounds to show that only finitely many words need to be considered [6]; the bound in the following theorem is due to Pappacena. [5]

Theorem 1 *Let complex n -by- n matrices A and B be given. The following are equivalent:*

- (a) *A and B are unitarily similar;*
- (b) *$\text{tr } W(A, A^*) = \text{tr } W(B, B^*)$ for every word $W(s, t)$ in two noncommuting variables;*
- (c) *$\text{tr } W(A, A^*) = \text{tr } W(B, B^*)$ for every word $W(s, t)$ in two noncommuting variables whose length is at most*

$$n\sqrt{\frac{2n^2}{n-1} + \frac{1}{4}} + \frac{n}{2} - 2.$$

3 A basic lemma

The key to obtaining our criteria for simultaneous unitary similarity and simultaneous unitary congruence is understanding the consequences of certain intertwining relations involving a special block matrix.

Lemma 2 *Consider the complex k -by- k block matrices*

$$A = \begin{bmatrix} 0 & I_n & A_{1,3} & \cdots & A_{1,k} \\ & 0 & I_n & \ddots & \vdots \\ & & 0 & \ddots & A_{k-2,k} \\ & & & \ddots & I_n \\ & & & & 0 \end{bmatrix} \quad (3)$$

and

$$B = \begin{bmatrix} 0 & I_n & B_{1,3} & \cdots & B_{1,k} \\ & 0 & I_n & \ddots & \vdots \\ & & 0 & \ddots & B_{k-2,k} \\ & & & \ddots & I_n \\ & & & & 0 \end{bmatrix} \quad (4)$$

in which every block is n -by- n . Define $A_{i,i+1} = B_{i,i+1} = I_n$ for all $i = 1, \dots, k-1$ and $A_{ij} = B_{ij} = 0$ whenever $i \geq j$. Let $W = [W_{ij}]_{i,j=1}^k$ be an nk -by- nk matrix that is partitioned conformally to A and B .

(a) Suppose that $AW = WB$. Then W is block upper triangular and $W_{11} = W_{22} = \dots = W_{kk}$.

(b) Suppose that W is unitary and $AW = WB$, that is, A and B are unitarily similar and $A = WBW^*$. Then $W_{11} = U$ is unitary and $W = U \oplus U \oplus \dots \oplus U$ is block diagonal. Moreover, $A_{ij} = UB_{ij}U^*$ for all i and j .

(c) Suppose that $A\bar{W} = WB$. Then W is block upper triangular, $W_{ii} = W_{11}$ if i is odd, and $W_{ii} = \overline{W_{11}}$ if i is even.

(d) Suppose that W is unitary and $A\bar{W} = WB$, that is, A is unitarily congruent to B and $A = WBW^T$. Then $W_{11} = U$ is unitary, $W_{ii} = U$ if i is odd, $W_{ii} = \bar{U}$ if i is even, and $W = U \oplus \bar{U} \oplus U \oplus \dots$ is block diagonal. Moreover,

- (d1) $A_{ij} = UB_{ij}U^*$ if i is odd and j is even;
- (d2) $A_{ij} = UB_{ij}U^T$ if i and j are both odd;
- (d3) $A_{ij} = \bar{U}B_{ij}U^*$ if i and j are both even; and
- (d4) $A_{ij} = \bar{U}B_{ij}U^T$ if i is even and j is odd.

Proof. A computation verifies the assertions in (a) and (c) about block triangularity: compare blocks in the respective identities $AW = WB$ and $A\bar{W} = WB$, starting in block position $(k, 1)$. Work to the right until reaching block position $(k, k-1)$. Move up to block position $(k-1, 1)$ and work to the right until reaching block position $(k-1, k-2)$. Repeat this process, moving up one block row at a time, until reaching block position $(2, 1)$.

The assertions in (a) and (c) about the main diagonal blocks of W follow (once one knows that W is block upper triangular) from comparing blocks in the respective identities in positions $(1, 2), \dots, (k-1, k)$.

The assertions in (b) and (d) about W reflect the facts that a block triangular unitary matrix is block diagonal, and the direct summands in a unitary direct sum are unitary. The asserted relationships between A_{ij} and B_{ij} follow (once one knows that W is block diagonal) from the respective identities $A_{ij}W_{jj} = W_{ii}A_{ij}$ and $A_{ij}\overline{W_{jj}} = W_{ii}A_{ij}$. ■

In summary, if the matrices A and B in (3) and (4) are unitarily similar then all of the pairs (A_{ij}, B_{ij}) are simultaneously unitarily similar. If A and B are unitarily congruent, then there is a single unitary matrix U involved in four types of unitary equivalence: certain pairs (A_{ij}, B_{ij}) are simultaneously

unitarily similar via U or \bar{U} , and certain pairs are simultaneously unitarily congruent via U or \bar{U} .

4 Simultaneous unitary similarity

It is useful to have an explicit statement of the criterion for simultaneous unitary similarity that is implicit in Lemma 2.

Theorem 3 *Let pairs $(A_1, B_1), \dots, (A_m, B_m)$ of n -by- n complex matrices be given. Choose k large enough so that the matrix A in (3) has at least m blocks above the second block superdiagonal. Place the matrices A_1, \dots, A_m in those blocks in any order, and place zero matrices in any unfilled blocks. Place B_1, \dots, B_m and zero matrices in corresponding blocks of the matrix B in (4). Then A is unitarily similar to B , if and only if the pairs $(A_1, B_1), \dots, (A_m, B_m)$ are simultaneously unitarily similar.*

For example, one could choose $k = m + 2$ and place the respective matrices of the pairs $(A_1, B_1), \dots, (A_m, B_m)$ in positions $(1, 3), \dots, (1, k)$ of the first block rows of (3) and (4), or in the blocks $(1, 3), (2, 4), \dots, (k - 2, k)$ of the third block superdiagonal of (3) and (4).

A natural extension of Specht's criterion to more than a single pair of matrices follows from the preceding theorem.

Corollary 4 *Given pairs $(A_1, B_1), \dots, (A_m, B_m)$ of n -by- n complex matrices are simultaneously unitarily similar if and only if $\text{tr } w(A_1, A_1^*, \dots, A_m, A_m^*) = \text{tr } w(B_1, B_1^*, \dots, B_m, B_m^*)$ for all words $w(s_1, t_1, \dots, s_m, t_m)$ in $2m$ noncommuting variables.*

Proof. Let $k = m + 2$ and consider a matrix A of the form (3) that is constructed by placing the matrices A_1, \dots, A_m sequentially in the m blocks of its third block superdiagonal, and placing zero blocks in all of its other blocks above the third block superdiagonal. Construct a matrix B of the form (4) in the same way using the matrices B_1, \dots, B_m .

Consider the following assertions:

- (a) $\text{tr } w(A_1, A_1^*, \dots, A_m, A_m^*) = \text{tr } w(B_1, B_1^*, \dots, B_m, B_m^*)$ for all words $w(s_1, t_1, \dots, s_m, t_m)$ in $2m$ noncommuting variables;
- (b) $\text{tr } W(A, A^*) = \text{tr } W(B, B^*)$ for every word $W(s, t)$ in two noncommuting variables;

- (c) A is unitarily similar to B ;
- (d) The pairs $(A_1, B_1), \dots, (A_m, B_m)$ are simultaneously unitarily similar.

It suffices to show that these four assertions are equivalent.

- (a) \Rightarrow (b) Each block of any word $W(A, A^*)$ is a linear combination of the identity matrix (we may think of it as an empty word) and words of the form $w(A_1, A_1^*, \dots, A_m, A_m^*)$.
- (b) \Rightarrow (c) Theorem 1.
- (c) \Rightarrow (d) Theorem 3.
- (d) \Rightarrow (a) The same computation that verified the identities (2). ■

Simultaneous unitary similarity of given pairs $(A_1, B_1), \dots, (A_m, B_m)$ of n -by- n complex matrices is equivalent to unitary similarity of two particular block matrices; Theorem 1 ensures that this latter unitary similarity can be confirmed or refuted with finitely many computations. Thus, there is a finite algorithm to determine whether a finite number of pairs of matrices are simultaneously unitarily similar.

Likewise, the trace criterion in Corollary 4 requires only finitely many computations: it requires verification only of enough identities of the form $\text{tr } w(A_1, A_1^*, \dots, A_m, A_m^*) = \text{tr } w(B_1, B_1^*, \dots, B_m, B_m^*)$ to ensure satisfaction of the finite number of identities of the form $\text{tr } W(A, A^*) = \text{tr } W(B, B^*)$ required by the Pappacena upper bound in Theorem 1.

5 Unitary congruence of a pair of matrices

Before we consider the role of Lemma 2 in assessing simultaneous unitary congruence and other simultaneous unitary equivalences, we need to consider the simplest case of unitary congruence of a single pair of matrices. If U is unitary and $A = UBU^T$, a calculation reveals that

$$AA^* = U(BB^*)U^*, \quad A\bar{A} = U(B\bar{B})U^*, \quad \text{and} \quad A^T\bar{A} = U(B^T\bar{B})U^*, \quad (5)$$

so three pairs of matrices related to A and B are simultaneously unitarily similar. Fortunately, this necessary condition is also sufficient:

Theorem 5 *Complex n -by- n matrices A and B are unitarily congruent if and only if the three pairs (AA^*, BB^*) , $(A\bar{A}, B\bar{B})$, and $(A^T\bar{A}, B^T\bar{B})$ are simultaneously unitarily similar. If either A or B is nonsingular, the third pair may be omitted.*

Proof. See [3]. ■

The preceding theorem and Theorem 3 imply the following criterion for unitary congruence.

Theorem 6 *Complex n -by- n matrices A and B are unitarily congruent if and only if the $4n$ -by- $4n$ matrices*

$$K_A = \begin{bmatrix} 0 & I_n & AA^* & A\bar{A} \\ & 0 & I_n & A^T \bar{A} \\ & & 0 & I_n \\ & & & 0 \end{bmatrix} \text{ and } K_B = \begin{bmatrix} 0 & I_n & BB^* & B\bar{B} \\ & 0 & I_n & B^T \bar{B} \\ & & 0 & I_n \\ & & & 0 \end{bmatrix} \quad (6)$$

are unitarily similar. If A or B is nonsingular, they are unitarily congruent if and only if

$$K'_A = \begin{bmatrix} 0 & I_n & AA^* & A\bar{A} \\ & 0 & I_n & 0 \\ & & 0 & I_n \\ & & & 0 \end{bmatrix} \text{ and } K'_B = \begin{bmatrix} 0 & I_n & BB^* & B\bar{B} \\ & 0 & I_n & 0 \\ & & 0 & I_n \\ & & & 0 \end{bmatrix} \quad (7)$$

are unitarily similar.

When one applies the criterion in Theorem 6, the bound in Theorem 1 ensures that it suffices to verify identities of the form

$$\text{tr } W(K_A, K_A^*) = \text{tr } W(K_B, K_B^*) \quad (8)$$

for all words $W(s, t)$ of length at most

$$4n\sqrt{\frac{32n^2}{4n-1} + \frac{1}{4}} + 2n - 2. \quad (9)$$

The matrices (6) and (7) are nilpotent of index four, so in verifying the trace identities (8) we need to consider only words, all of whose factor exponents are three or less.

For $n = 2$, the bound (9) says that it suffices to consider all words of length at most 37, so a great many words must be considered, even in the smallest case. For $n = 3$ the upper bound on the length is 66; for $n = 4$ it is 116. In contrast, when employing Specht's criterion to test a pair of 2-by-2 matrices for unitary similarity, it suffices to check traces of only three words

of length at most two; to test a pair of 3-by-3 matrices it suffices to check traces of only seven words of length at most six. It is not known whether the special form of the matrices in (6) and (7), some special features in low dimensional cases, or some clever insight would permit the upper bound (9) to be reduced significantly.

6 Simultaneous unitary equivalences

We now make explicit the solution of the General Problem that is implicit in Lemma 2.

Theorem 7 *Let m_1, m_2, m_3 , and m_4 be given nonnegative integers. Let*

$$\begin{aligned} (A_1^{(1)}, B_1^{(1)}), \dots, (A_{m_1}^{(1)}, B_{m_1}^{(1)}), & \quad (A_1^{(2)}, B_1^{(2)}), \dots, (A_{m_2}^{(2)}, B_{m_2}^{(2)}), \\ (A_1^{(3)}, B_1^{(3)}), \dots, (A_{m_3}^{(3)}, B_{m_3}^{(3)}), & \quad (A_1^{(4)}, B_1^{(4)}), \dots, (A_{m_4}^{(4)}, B_{m_4}^{(4)}) \end{aligned}$$

be given pairs of n -by- n complex matrices. Choose k large enough so that the matrix A in (3) has enough blocks above the second block superdiagonal to accommodate the following construction:

- (1) Place the matrices $A_1^{(1)}, \dots, A_{m_1}^{(1)}$ (in any desired order) in (i, j) blocks of A such that i is odd, j is even, and $j - i \geq 2$; place the matrices $B_1^{(1)}, \dots, B_{m_1}^{(1)}$ in corresponding positions in B .*
- (2) Place $A_1^{(2)}, \dots, A_{m_2}^{(2)}$ (in any desired order) in (i, j) blocks of A such that i and j are both odd and $j - i \geq 2$; place $B_1^{(2)}, \dots, B_{m_2}^{(2)}$ in corresponding positions in B .*
- (3) Place $A_1^{(3)}, \dots, A_{m_3}^{(3)}$ (in any desired order) in (i, j) blocks of A such that i and j are both even and $j - i \geq 2$; place $B_1^{(3)}, \dots, B_{m_3}^{(3)}$ in corresponding positions in B .*
- (4) Place $A_1^{(4)}, \dots, A_{m_4}^{(4)}$ (in any desired order) in (i, j) blocks of A such that i is even, j is odd, and $j - i \geq 2$; place $B_1^{(4)}, \dots, B_{m_4}^{(4)}$ in corresponding positions in B .*
- (5) Place zero matrices in any unfilled blocks of A and B .*

Then A is unitarily congruent to B if and only if there is a unitary matrix U such that

- (1') $A_i^{(1)} = UB_i^{(1)}U^*$ for all $i = 1, \dots, m_1$;*
- (2') $A_i^{(2)} = UB_i^{(2)}U^T$ for all $i = 1, \dots, m_2$;*
- (3') $A_i^{(3)} = \bar{U}B_i^{(3)}U^*$ for all $i = 1, \dots, m_3$; and*
- (4') $A_i^{(4)} = \bar{U}B_i^{(4)}U^T$ for all $i = 1, \dots, m_4$.*

Suppose that four sets of pairs of n -by- n complex matrices are given (some of these sets may be empty), and it is required to determine if the simultaneous unitary equivalences stated in the General Problem are valid. Our algorithm proceeds as follows:

Algorithm 8 *Construct the block matrices A and B according to the prescription in the preceding theorem. Then construct*

$$K_A = \begin{bmatrix} 0 & I_n & AA^* & A\bar{A} \\ & 0 & I_n & A^T \bar{A} \\ & & 0 & I_n \\ & & & 0 \end{bmatrix} \text{ and } K_B = \begin{bmatrix} 0 & I_n & BB^* & B\bar{B} \\ & 0 & I_n & B^T \bar{B} \\ & & 0 & I_n \\ & & & 0 \end{bmatrix}.$$

The four given sets of pairs of matrices satisfy the required simultaneous unitary equivalences if and only if K_A and K_B are unitarily similar. That unitary similarity can be confirmed or refuted with finitely many computations by using Theorem 1. In those computations, only words with factor exponents at most 3 need to be considered.

7 Some comments on previous work

A criterion for simultaneous unitary similarity that reduces the problem to one of verifying unitary similarity of a single pair of block matrices is in Section 2.3 of the 1998 paper [8]. We employ different block matrices in Lemma 2 because we want it to embrace both simultaneous unitary congruence and simultaneous unitary similarity.

The problem of simultaneous unitary similarity was also studied in the 2003 paper [1]. The criterion developed there is formally equivalent to the finite version of our Corollary 4.

The recent paper [2] makes the important observation that the criterion in [3] can be combined with a test for simultaneous unitary similarity to give a criterion for unitary congruence that can be verified with finitely many computations; the authors use the test in [1]. However, because of an extra condition imposed in [1] on matrix families whose simultaneous unitary similarity is to be tested, the criterion in [2] for determining unitary congruence of two general matrices requires assessing simultaneous unitary similarity of the *four* pairs (AA^*, BB^*) , $(A\bar{A}, B\bar{B})$, $(A^T A^*, B^T B^*)$, and $(A^T \bar{A}, B^T \bar{B})$ rather than the *three* pairs in (5).

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